JHEP HYPER VERSION

Received: *May 2, 2001*, Accepted: *June 9, 2001*

On free energy of 2-d black hole in bosonic string theory

Vladimir A. Kazakov

Laboratoire de Physique
Théorique de l'Ecole Normale Supérieure
E-mail: vladimir.kazakov@lpt.ens.fr

Arkady A. Tseytlin*

24 rue Lhomond, 75231 Paris Cedex, France, and
Physics Department, Ohio State University
Columbus, OH43210-1106, USA
E-mail: tseytlin.1@osu.edu

Abstract: Trying to interpret recent matrix model results (hep-th/0101011) we discuss computation of classical free energy of exact dilatonic 2-d black hole from the effective action of string theory. The euclidean space-time action evaluated on the black hole background is divergent due to linear dilaton vacuum contribution, and its finite part depends on a subtraction procedure. The thermodynamic approach based on subtracting the vacuum contribution for fixed values of temperature and dilaton charge at the "wall" gives (as in the leading-order black hole case) $S = M/T$ for the entropy and zero value for the free energy F. We suggest that in order to establish a correspondence with a non-vanishing matrix model result for F one may need an alternative reparametrization-invariant subtraction procedure using analogy with non-critical string theory (i.e. replacing the spatial coordinate by the dilaton field). The subtraction of the dilaton divergence then produces a finite value for the free energy. We also propose a microscopic estimate for the entropy and energy of the black hole based on the contribution of non-singlet states of the matrix model.

Keywords: Conformal Field Models in String Theory, Black Holes in String Theory, 2D Gravity, Matrix Models.

*Also at Imperial College, London and Lebedev Physics Institute, Moscow.

Contents

1. Introduction **1**

2. Bulk and boundary terms in string theory effective action **3**

3. Free energy of exact 2-d black hole background **6**
3.1 Thermodynamic "subtraction at the wall" approach 9
3.2 "Dilaton subtraction" approach 11
3.3 Alternative subtractions 12

4. Free energy in matrix model of 2-d black hole **13**
4.1 Noncritical $c_M = 1$ string theory and free energy from matrix model 13
4.2 Comments on interpretation of the matrix model result 15

5. Concluding remarks **18**

A. Hamiltonian of the matrix model in non-singlet representation **19**

1. Introduction

Recent investigation [2] suggested a way to compute free energy of the dilatonic 2-d black hole [3, 4] using matrix model methods. It is of obvious interest to try to compare the matrix model prediction with the free energy which follows from the string low-energy space-time effective action.

The previous analysis of thermodynamics of euclidean black hole [1] used the leading-order (1-loop) background of [4] with the conclusion that the entropy is $S = M/T$, implying that the black-hole free energy $F = E - ST$ is zero (equivalent results were found in [5, 6, 7]). Here $E = M$ is the leadeing order black hole mass (equal to its ADM mass [3, 1, 5]), i.e. the full energy minus the divergent dilaton vacuum contribution.

At the same time, the bosonic matrix model calculation [2] gave a non-zero result

$$-\mathcal{F}_0 = \beta F \propto e^{-2\phi_0} \tag{1.1}$$

for the tree-level part of the string theory partition function. One may try to reconcile these facts by acknowledging that since the full string-theory expression contains the divergent dilaton vacuum contribution, a finite (subtracted) value of free energy may be ambiguous, i.e. non-universal [2]. The finite matrix model result was obtained by solving Toda equation with certain natural boundary conditions. On string theory (effective gravitational field theory) side, one finds a finite (zero) value for F after a particular (thermodynamic ensemble motivated) subtraction of the dilaton vacuum contribution.

The comparison between string theory and matrix model results in [2] was referring to the analysis of 2-d black hole thermodynamics for the leading-order form of the black hole background. One may expect that the effective field theory result $F = 0$ may change if one considers the exact (all-order in α') form [8] of the black hole background. For example, one may finish with $S = \nu M/T$, where ν is a numerical coefficient different from one, leading to $F \neq 0$.

The aim of the present paper is to address this issue. First, we shall repeat the thermodynamic analysis of [1] replacing the leading-order black hole background by its exact form. We shall show that the subtracted black-hole part of the free energy is still zero. Our discussion will highlight several important open questions, in particular, the origin of boundary terms in string theory effective action and the meaning and consistent implementation of subtraction of the dilaton vacuum contribution.

The approach of [1] to computing the free energy is based on the assumption that one can prepare the canonical ensemble (for the black hole in equilibrium with radiation) with any values of the temperature and dilaton charge at the boundary. At the same time, the value of the temperature of the 2-d dilatonic black hole background in string theory is not a free parameter being fixed by the central charge condition. The thermodynamic approach is thus implicitly assuming that this constraint can be relaxed, i.e. that the temperature can be made arbitrary by adding extra matter fields.

This approach may not, however, be an adequate one for comparison with the matrix model. We shall suggest that in this case one should use a different subtraction procedure in defining the effective action. As a result, we shall find a non-zero value for the string partition function on 2-sphere consistent with the matrix model prediction [2].

While in the case of the fermionic string (with $N = 1$ world-sheet supersymmetry) the leading-order 2-d black hole solution of [4, 3] is exact, it receives, in general, α'-corrections in the bosonic case. Below (in section 3) we shall reconsider the computation of the classical contribution to the black hole free energy using the α'-corrected form of the background [8] (see also [9, 10]). This background is the exact solution of the conformal invariance equations to all orders in sigma model loop expansion in the standard (dimensional regularization with minimal subtraction) scheme [11, 12, 13].[1]

[1] There exists a formal scheme in which the leading-order solution remains exact [13, 14], but it

We shall follow closely the discussion in [1] (see also [5, 6]). Their results showed analogy with similar results in four dimensions [15] but the presence of the dilaton field introduced important subtleties.

The question of computing the entropy for the exact black hole background of [8] was raised already in [1], but it was not clear how to do that computation given that the free energy should be determined by the value of the full euclidean string action I (with unknown higher order α'^n corrections) evaluated on the exact solution.[2]

In spite of the fact that the higher order α' terms in I are not known in general, one of our main observations is that the value of the full action *at the exact conformal point*, i.e. its exact extremum, may be computed, provided one makes a natural assumption about the structure of the boundary terms in the action. As we shall argue in section 2, the (properly defined) volume term in the effective action always vanishes at its exact extremum, and there exists a natural simple choice for the boundary term.

In section 3 we shall first review the structure of the exact 2-d black hole background [8] and then compute the value of the boundary term in the action on this background. In section 3.1 we shall find the corresponding free energy F and entropy S by introducing an IR cutoff ("position of a wall") and treating, following [1], the dilaton charge and the temperature at the wall as the arguments of F. Subtracting the dilaton vacuum contribution from the energy and thus getting the black hole mass M we shall then find that $S = M/T$, implying that the free energy vanishes. We shall then explain in section 3.2 an alternative (non-critical string theory motivated) subtraction procedure for the string effective action on the black hole background. It leads to a finite value of the action which is qualitatively consistent with the expression in [2]. Alternative ways of doing the infinite subtraction of the dilaton vacuum contribution will be discussed in section 3.3.

In section 4 we shall review the matrix model result of [2] for the free energy in the tree approximation and try to extract from it the energy and the entropy of the black hole (related matrix model hamiltonian will be given in appendix).

Some concluding remarks will be made in section 5.

2. Bulk and boundary terms in string theory effective action

The standard way to compute a classical entropy associated with a gravitational background is to evaluate the euclidean effective action on this solution [15]. One follows the analogy with path integral approach to quantum gravity where the partition function is given by the integral over the space of all field configurations that are periodic in euclidean time. The classical contribution is thus $e^{-I} = e^{-\beta F}$.

is the exact background which is "seen" by a string (tachyon) probe [8].

[2]For discussions of black hole entropy in higher derivative gravity theories see, e.g., [16, 17, 18, 19].

The string effective action for "massless" modes should be given, in general, by the sum of a bulk term $I_{\mathcal{M}}$ and a boundary term $I_{\partial\mathcal{M}}$. Evaluating it on a solution following [15] one gets the free energy

$$F = T\, I\ , \qquad I = I_{\mathcal{M}} + I_{\partial\mathcal{M}}\ , \qquad T = \beta^{-1}\,. \tag{2.1}$$

Both terms in I are expected to be infinite series in α' and thus are not known exactly.

We shall by-pass this problem by arguing that a properly defined bulk term $I_{\mathcal{M}}$ should actually vanish when evaluated on the exact conformal background, while the corresponding boundary contribution $I_{\partial\mathcal{M}}$ in this case should be given just by the two standard lowest-derivative metric and dilaton dependent terms. If the explicit all-order form of a conformal background is known, one is then able to get the exact expression for the free energy F just by evaluating the two leading boundary terms in I on this background.

Let us consider first the bulk term $I_{\mathcal{M}}$. To obtain the effective action for the massless closed string metric and dilaton fields $I[G,\phi]$ from the string partition function Z (generating functional for correlators of vertex operators [20]) one needs to subtract both the massless poles (UV logarithms) [21] and the SL$(2,\mathbb{C})$ Möbius infinities [22]. While the former are power-like in the open string case [23, 24], they are also logarithmic in the closed string case [24, 22], implying that an "extra" log should be subtracted from Z. The RG invariant way to do that subtraction is to apply $\partial/\partial \ln\epsilon$ to the bare value of Z [22]. Expressed in terms of renormalized couplings (tuning the renormalized value of the tachyon coupling to zero) the effective action is then given by [22, 25] ($\lambda^i = (G,\phi)$)

$$I_{\mathcal{M}} = -\left(\frac{\partial Z}{\partial \ln\epsilon}\right)_{\epsilon=1} = \beta^i \cdot \frac{\delta Z}{\delta\lambda^i}\,, \tag{2.2}$$

where the objects in the second equality are assumed to be expressed in terms of the renormalized couplings. One can then argue that there exists a scheme where [22, 25]

$$I_{\mathcal{M}} = -\left(\frac{\partial V}{\partial \ln\epsilon}\right)_{\epsilon=1}\ , \qquad V = \kappa_0 \int_{\mathcal{M}} d^D x\ \mathcal{V}\,, \qquad \mathcal{V} \equiv \sqrt{G}\ e^{-2\phi}\,. \tag{2.3}$$

Here G and ϕ in $\mathcal{V}$ are the running (cut-off dependent) couplings. Using the diffeomorphism invariance of Z, the expression for $I_{\mathcal{M}}$ can be rewritten also as $I_{\mathcal{M}} = \bar\beta^i \cdot \frac{\delta Z}{\delta\lambda^i}$, where $\bar\beta^i = \beta^i + (\delta\lambda^i)_{\alpha'\partial\phi}$ are the Weyl-anomaly coefficients, which differ from the β-functions by the diffeomorphism terms, i.e. $\bar\beta^\phi = \beta^\phi + \alpha'\partial^m\phi\partial_m\phi$, $\bar\beta^G_{mn} = \beta^G_{mn} + 2\alpha'\mathcal{D}_m\mathcal{D}_n\phi$. Explicitly ($\kappa_0 \sim \alpha'^{-D/2}$)

$$I_{\mathcal{M}} = \kappa_0 \int_{\mathcal{M}} d^D x\, \sqrt{G}\, e^{-2\phi} \left(2\bar\beta^\phi - \frac{1}{2} G^{mn}\bar\beta^G_{mn}\right) = 2\kappa_0 \int_{\mathcal{M}} d^D x\, \sqrt{G}\, e^{-2\phi}\, \tilde\beta^\phi\,, \tag{2.4}$$

where the "central charge lagrangian"

$$\tilde{\beta}^{\phi} = \bar{\beta}^{\phi} - \frac{1}{4} G^{mn} \bar{\beta}^{G}_{mn} \tag{2.5}$$

is the total conformal anomaly of the 2-d σ-model (coinciding with the central charge at the conformal point): $\langle T^a_a \rangle = \frac{1}{4\pi} \tilde{\beta}^{\phi} R^{(2)} + O(\bar{\beta}^G)$. The leading terms in its derivative expansion are

$$\tilde{\beta}^{\phi} = c_0 - \frac{1}{4}\alpha'(R + 4D^2\phi - 4\partial_m\phi\partial^m\phi) + O(\alpha'^2 R^2_{mnkl}) \, , \qquad c_0 = \frac{1}{6}(D-26) \, . \tag{2.6}$$

The resulting action reproduces the Weyl-invariance conditions $\bar{\beta}^i = 0$ as its equations of motion [26].

Even though we do not know the explicit form of the "central charge" $\tilde{\beta}^{\phi}$ to all orders in α', we can still deduce that since $\tilde{\beta}^{\phi} = 0$ is one of the conditions of 2-d Weyl invariance (dilaton equation of motion), the bulk term in the action given by (2.4) and (2.6) *vanishes* on any exact string solution.

This is the expected conclusion: the partition function of a 2-d CFT should not depend on a cutoff, and thus its derivative in (2.2) (or, equivalently, its ratio with the infinite volume of the SL$(2,\mathbb{C})$ Möbius group one needs to divide over to obtain the *string theory* partition function on S^2) should *vanish* at a conformal point.

Let us now turn to the boundary contribution $I_{\partial\mathcal{M}}$. While we do not know how to derive it systematically from the string σ-model path integral, one can try to fix it using indirect considerations. The two leading field-dependent terms in (2.6) — R and $D^2\phi$ — contain second derivatives, so we may follow the standard logic of "integrating by parts", i.e. adding the corresponding boundary terms to have a well-posed variational principle that involves only the fields and their first derivatives [27, 15]. That gives

$$I_{\partial\mathcal{M}} = -2\alpha'\kappa_0 \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{\gamma}\, e^{-2\phi}(K - 2\partial_n\phi) = -2\alpha'\kappa_0 \int_{\partial\mathcal{M}} d^{D-1}x\sqrt{\gamma}\,\nabla_a(e^{-2\phi}n^a) \, , \tag{2.7}$$

where $\partial_n = n^a\nabla_a$ (n^a is the unit outward normal vector to the boundary), K is the second fundamental form of the boundary $\nabla_a n^a$ and γ is the induced metric. For a diagonal metric of the type we shall consider below this can be put in the following simple form

$$I_{\partial\mathcal{M}} = -2\alpha'\kappa_0 \int_{\partial\mathcal{M}} d^{D-1}x\, \partial_n(\sqrt{G}e^{-2\phi}) = -2\alpha'\kappa_0 \int_{\partial\mathcal{M}} d^{D-1}x\, \partial_n\mathcal{V} \, , \tag{2.8}$$

where $\mathcal{V}$ is the same volume density as in (2.3).

Naively, one could think that the presence of higher-order terms in the beta functions and the effective action implies that one should also add higher-derivative

JHEP06(2001)021

terms to the boundary part of the action. However, in general it is not possible to derive such terms just from the requirement of having a consistent initial value problem [28].[3]

Our main assumption will be that (at least at the conformal point) (2.7) represents the *exact* expression for the boundary term in the tree-level string effective action. This is a natural assumption for the following reason. If one defines the string effective action by reconstructing it from the string scattering amplitudes on an asymptotically flat space-time, one needs to properly define the 2-nd derivative (propagator) terms in the action — by adding the boundary terms (2.7).[4] One may then think that all higher-order corrections (that can be always arranged so that they do not change the propagators of the massless fields [30, 31]) should represent interaction vertices (treated perturbatively in α') and thus should not be accompanied by boundary terms.

3. Free energy of exact 2-d black hole background

The 2-d black hole background [4] was shown in [3] to be described by exact CFT represented by gauged $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ WZW model. The exact background as "seen" by a local fundamental string probe (tachyon field) was extracted by [8] from the hamiltonian of this CFT interpreted as a generalized laplacian on the coset space.[5] It was checked that this background is indeed a solution of the $\bar{\beta}^i = 0$ equations at the 2-loop [13] and 3-loop [12] levels (with the beta-function computed in the standard dimensional regularization with minimal subtraction scheme).

The expression for a background extracted from a gauged WZW model is, in general, scheme-dependent [13, 14]. There exist a non-standard "leading-order scheme" in which the background has its leading-order 1-loop form (but the tachyon beta-function is modified by α' corrections) and a "CFT scheme" where (in bosonic case) the background gets corrections to all orders in α' (but the tachyon equation retains

[3] In the standard QFT context higher-derivative boundary terms are indeed induced in the quantum effective action if one quantizes matter fields coupled to gravity on a manifold with boundary. Heuristically, one could try, by analogy with how that is done for the Seeley coefficients of laplacians, to derive the boundary terms which are associated with higher-order ($R^n + \cdots$) terms in $I_{\mathcal{M}}$ by replacing $\mathcal{M}$ by its double with no boundary, and then computing $I_{\mathcal{M}}$ for the fields on the double that will have delta-function-type singularities along the joining (boundary) submanifold, producing boundary terms as a result. However, there does not seem to be an intrinsic reason for this procedure in the present context. Again, the idea of reconstructing boundary terms from a well-definiteness of variational problem does not seem to apply to higher-order terms (one would in any case need to fix all higher derivatives of variation of the fields at the boundary).

[4] Note that the classical graviton S-matrix in the Einstein theory is indeed generated by the boundary term K in (2.7) evaluated on asymptotic wave $h^{(in)}$ dependent solution of the Einstein equations [29].

[5] The same background can be obtained also directly in the σ-model framework by integrating out 2-d gauge fields [9, 10] (see [14] for a review).

JHEP06(2001)021

its leading-order form).[6] The two backgrounds describe the same string geometry in the sense that since a local probe of the geometry is the tachyon field, and the tachyon equation remains exactly the same differential equation as implied by the coset CFT (even though it looks different when expressed in terms of different couplings G and ϕ).

The euclidean 2-d black hole background in the leading-order scheme [4, 3]

$$ds^2_{\rm lead} = dx^2 + \tanh^2 bx\, d\tau^2\,, \qquad \phi_{\rm lead} = \phi_0 - \ln\cosh bx\,, \tag{3.1}$$

is related to the "exact" background in the CFT scheme [8][7]

$$ds^2 = dx^2 + \frac{\tanh^2 bx}{1 - p\tanh^2 bx} d\tau^2\,, \tag{3.2}$$

$$\phi = \phi_0 - \ln\cosh bx - \frac{1}{4}\ln(1 - p\tanh^2 bx)\,, \tag{3.3}$$

$$\alpha' b^2 = \frac{1}{k-2}\,, \qquad p \equiv \frac{2\alpha' b^2}{1 + 2\alpha' b^2} = \frac{2}{k}\,, \qquad D - 26 + 6\alpha' b^2 = 0\,, \tag{3.4}$$

by the following local covariant redefinition [13]

$$(G_{mn})_{\rm lead} = G_{mn} - \frac{2\alpha'}{1 + \frac{1}{2}\alpha' R}[\partial_m\phi\partial_n\phi - (\partial\phi)^2 G_{mn}]\,, \tag{3.5}$$

$$\phi_{\rm lead} = \phi - \frac{1}{4}\ln\left(1 + \frac{1}{2}\alpha' R\right)\,.$$

Note also that[8]

$$\sqrt{G_{\rm lead}}\; e^{-2\phi_{\rm lead}} = \sqrt{G}\; e^{-2\phi}\,. \tag{3.6}$$

$e^{\phi_0} = g_s$ is the string coupling at the horizon $x = 0$ (tip of the "cigar").

To find the free energy we shall use the exact form of the background (3.2). Since $\tilde\beta^\phi$ (computed in the standard scheme) vanishes for the exact background (3.2) and (3.3), the bulk term in the action (2.4) is zero, and thus the free energy (2.1) is determined by the boundary term (2.7). In view of the representation (2.8) and the equality of the two volume density factors (3.6), the boundary contribution (2.7) evaluated for the exact background will look formally the same as for the leading solution computed in [1]. However, the thermodynamic properties of the two backgrounds differ since they have different large distance asymptotics.

[6]Note that in 2 dimensions one can always choose a scheme such that any solution of the effective equations has its leading-order form [32].

[7]In our normalization the σ-model action is: $I = \frac{1}{4\pi\alpha'}\int d^2z\sqrt{g}[\partial_a x^m \partial^a x^n G_{mn}(x) + \alpha' R^{(2)}\phi(x)]$. The coordinates that naturally appear in the WZW context are: $r = bx$, $\varphi = b\tau$, $\alpha' b^2 = \frac{1}{k-2}$.

[8]Such equality of the measure factors is true in general for backgrounds obtained from gauged WZW models [9, 10].

The metrics (3.1) and (3.2) are of the form

$$ds^2 = dx^2 + f^2(x) d\tau^2 \,, \tag{3.7}$$

for which (2.7) reduces to (2.8) and thus gives (in what follows we set $\alpha' \kappa_0 = 1$)

$$I_{\partial \mathcal{M}} = -2 \int_0^{\beta_0} d\tau \partial_x (f e^{-2\phi})|_{x=x_{\rm bndry}} \,. \tag{3.8}$$

As a result, we get the same-looking expression for the action as found in [1] for (3.1)

$$I_{\partial \mathcal{M}} = -2\beta_0 b e^{-2\phi_0} \cosh 2b x_{\rm bndry} \,. \tag{3.9}$$

Here β_0 is the period of τ which is determined from the condition of the absence of conical singularity near $x = 0$.[9] From (3.3) we get (taking $x \to 0$)

$$T_0 = \beta_0^{-1} = \frac{b}{2\pi} = \frac{1}{2\pi \sqrt{\alpha'} \sqrt{k-2}} \,. \tag{3.10}$$

As a consequence, (3.9) becomes

$$I_{\partial \mathcal{M}} = -4\pi e^{-2\phi_0} \cosh 2b x_{\rm bndry} \,. \tag{3.11}$$

Note that while for the leading-order background (3.2) $f(x) \to 1$ for large x and thus the asymptotic temperature is the same as T_0, this is not true for the exact background (3.2): here $f(x) \to (1-p)^{-1/2}$ and thus the Hawking temperature is [33][10]

$$\begin{aligned} T = \beta^{-1} &= \frac{b}{2\pi}(1-p)^{1/2} = T_0 (1-p)^{1/2} \\ &= \frac{T_0}{\sqrt{1 + 8\pi^2 \alpha' T_0^2}} = \frac{1}{2\pi \sqrt{\alpha'} \sqrt{k}} = \frac{\sqrt{p}}{2\pi \sqrt{2} \sqrt{\alpha'}} \,. \end{aligned} \tag{3.12}$$

Below we shall keep the dependence on the parameter p in (3.2) and (3.3) explicit so that the quantities corresponding to the leading-order black hole can be recovered by setting $p = 0$ (see (3.4)). Note that if we put $D = 2$ in (3.4) we get $\alpha' b^2 = 4$, i.e. $k = 9/4$ [3] and thus the temperature will not be a free parameter. It is possible to assume instead that the temperature may be changed by changing the effective central charge as a result of varying the number $N (= D - 2)$ of extra massless "matter" fields that can be added to the system.

[9]As usual, the flat space has topology of a cylinder, $S^1 \times R^1$, where the euclidean time is periodic with an arbitrary period, while the euclidean black hole has topology of a hemisphere, so its local temperature is given by the inverse proper periodicity of the euclidean time needed to obtain a regular disc metric near the tip.

[10]One could, of course, avoid introducing T_0 by rescaling τ in the metric (3.2) so that to make (3.7) having the canonical flat space form at infinity with $f \to 1$. Then T would be the inverse period of the euclidean time.

JHEP06(2001)021

3.1 Thermodynamic "subtraction at the wall" approach

As usual, we shall restrict x to the interval $0 \le x < \infty$. Following [1], let us compute all thermodynamic quantities not directly at the boundary $x = \infty$ but at a finite distance $x = x_{\rm w}$ ("position of a wall") treating this as an IR regularization. We shall then subtract the divergent contribution of the asymptotic cylinder. This regularization is needed since one does not have a translationally invariant vacuum at infinity: there is a linearly growing dilaton.

The value of the temperature at the wall is determined by the periodicity of the effective euclidean time coordinate there, i.e. [1] (τ in (3.2) has the period β_0, see (3.10))

$$T_{\rm w} = T(x_{\rm w}) = T_0 f^{-1}(x_{\rm w}) = T_0(\coth^2 bx_{\rm w} - p)^{1/2}\,, \tag{3.13}$$

or, equivalently,

$$T_{\rm w} = T\left(\frac{\coth^2 bx_{\rm w} - p}{1-p}\right)^{1/2}, \qquad T = T_{\rm w}(x_{\rm w} = \infty)\,. \tag{3.14}$$

Note that as in the case of the leading-order background with $p = 0$ one has $T_{\rm w} \ge T$.

As in [1] we shall view the values of the temperature $T_{\rm w} = T(x_{\rm w})$ and the dilaton $\phi_{\rm w} = \phi(x_{\rm w})$ as the basic free parameters of the thermodynamic system.

We define the dilaton charge as in [1]

$$\mathcal{D}_{\rm w} = e^{-2\phi_{\rm w}} = e^{-2\phi_0}\cosh^2 bx_{\rm w}(1 - p\tanh^2 bx_{\rm w})^{1/2}\,. \tag{3.15}$$

Then the free energy at the wall that follows from (3.9) is

$$\begin{aligned} F_{\rm w} &= T_{\rm w} I_{\partial\mathcal{M}}(x_{\rm w}) = -4\pi T_{\rm w} e^{-2\phi_0}\cosh 2bx_{\rm w} \\ &= -4\pi\mathcal{D}_{\rm w}\left(\sqrt{T_{\rm w}^2 + pT_0^2} + \frac{T_0^2}{\sqrt{T_{\rm w}^2 + pT_0^2}}\right) \equiv F(\mathcal{D}_{\rm w}, T_{\rm w})\,. \end{aligned} \tag{3.16}$$

For $p = 0$ this reduces to the expression for the leading-order background (3.1) in [1]. Note that the free energy is divergent for $x_{\rm w} \to \infty$

$$F_{\rm w} = -2\pi T e^{-2\phi_0}[e^{2bx_{\rm w}} + 2(1-p)^{-1} + O(e^{-bx_{\rm w}})]\,. \tag{3.17}$$

One may try to subtract the divergence in a naive way, getting a *non-zero* result for the free energy (but zero value of the energy, see subsection 3.3 below). However, such a direct subtraction of the $e^{2bx_{\rm w}}$ term is problematic since it is coordinate-dependent.

One systematic approach is to follow [1, 6] and to subtract the flat space vacuum contribution to the free energy at fixed values of the basic variables $T_{\rm w}$ and $\mathcal{D}_{\rm w}$. The flat space metric is (3.7) with $f = 1$ and $\phi = \phi_0' - bx$, so that (3.8) gives[11]

[11] We take the flat space metric as $dx^2 + d\tau'^2$ where τ' has periodicity $\beta_{\rm w}$. An equivalent approach is to take it as the asymptotic form of (3.2), $dx^2 + (1-p)^{-1}d\tau^2$ with periodicity of $(1-p)^{-1/2}\tau$ still being $\beta_{\rm w}$ to match the definition of temperature in (3.13). Note also that, as in [1], we ignore the contribution of the other boundary of the flat-space vacuum cylinder.

JHEP06(2001)021

$I^{(\mathrm{vac})}_{\partial\mathcal{M}} = -4\beta_{\mathrm{w}} b\mathcal{D}_{\mathrm{w}}$, where we took the period of τ to be β_{w} and replaced $e^{-2\phi}$ by $\mathcal{D}_{\mathrm{w}}$. Then

$$F^{(\mathrm{vac})}_{\mathrm{w}} = T_{\mathrm{w}} I^{(\mathrm{vac})}_{\partial\mathcal{M}} = -4b\mathcal{D}_{\mathrm{w}} = -8\pi T_0 \mathcal{D}_{\mathrm{w}} \,, \tag{3.18}$$

so that the subtracted value of the free energy is

$$F'_{\mathrm{w}} = F_{\mathrm{w}} - F^{(\mathrm{vac})}_{\mathrm{w}} = -4\pi\mathcal{D}_{\mathrm{w}} \left(\sqrt{T^2_{\mathrm{w}} + pT^2_0} + \frac{T^2_0}{\sqrt{T^2_{\mathrm{w}} + pT^2_0}} - 2T_0 \right) . \tag{3.19}$$

Expanding this at $x_{\mathrm{w}} \to \infty$ we find

$$F'_{\mathrm{w}} = -4\pi T e^{-2\phi_0} e^{-2bx_{\mathrm{w}}} + O(e^{-4bx_{\mathrm{w}}}) \,, \qquad \text{i.e. } F'_{\infty} = 0 \,. \tag{3.20}$$

Thus the subtracted value of the free energy *vanishes*.

The basic thermodynamic relations are

$$S_{\mathrm{w}} = -\left(\frac{\partial F_{\mathrm{w}}}{\partial T_{\mathrm{w}}}\right)_{\mathcal{D}_{\mathrm{w}}} \,, \qquad \psi_{\mathrm{w}} = -\left(\frac{\partial F_{\mathrm{w}}}{\partial \mathcal{D}_{\mathrm{w}}}\right)_{T_{\mathrm{w}}} \,, \tag{3.21}$$

$$E_{\mathrm{w}} = F_{\mathrm{w}} + S_{\mathrm{w}} T_{\mathrm{w}} \,, \tag{3.22}$$

where S is the entropy, ψ is the chemical potential associated with the dilaton charge and E is the total energy. From (3.18)

$$S^{(\mathrm{vac})}_{\mathrm{w}} = 0 \,, \qquad E^{(\mathrm{vac})}_{\mathrm{w}} = F^{(\mathrm{vac})}_{\mathrm{w}} = -4b\mathcal{D}_{\mathrm{w}} \,. \tag{3.23}$$

Differentiating (3.16) over T_{w} we find

$$S_{\mathrm{w}} = 4\pi\mathcal{D}_{\mathrm{w}} \left(1 + p\frac{T^2_0}{T^2_{\mathrm{w}}}\right)^{-3/2} \left[1 - (1-p)\frac{T^2_0}{T^2_{\mathrm{w}}}\right] = 4\pi\mathcal{D}_{\mathrm{w}} \left(1 + \frac{p}{1-p}\frac{T^2}{T^2_{\mathrm{w}}}\right)^{-3/2} \left(1 - \frac{T^2}{T^2_{\mathrm{w}}}\right) . \tag{3.24}$$

While the entropy was constant for the leading $p = 0$ solution [1], this is no longer so for the exact solution. Explicitly, in terms of x_{w}

$$S_{\mathrm{w}} = 4\pi e^{-2\phi_0} (1 - p \tanh^2 bx_{\mathrm{w}}) \,, \tag{3.25}$$

so that the entropy is positive as long as $p < 1$, i.e. $k - 2 > 0$. Its asymptotic $x_{\mathrm{w}} \to \infty$ boundary value is[12]

$$S = 4\pi e^{-2\phi_0} (1 - p) \,. \tag{3.26}$$

For the energy (3.22) we find

$$\begin{aligned} E_{\mathrm{w}} &= -4\pi\mathcal{D}_{\mathrm{w}} \frac{T^2_0}{T_{\mathrm{w}}} \left(1 + p\frac{T^2_0}{T^2_{\mathrm{w}}}\right)^{-3/2} \left[2 + p + p(p+1)\frac{T^2_0}{T^2_{\mathrm{w}}}\right] \\ &= -4\pi e^{-2\phi_0} T_0 (\sinh 2bx_{\mathrm{w}} + p \tanh bx_{\mathrm{w}})(1 - p\tanh^2 bx_{\mathrm{w}})^{1/2} \,. \end{aligned} \tag{3.27}$$

[12] It is non-zero since the vanishing of the factor $1 - T^2/T^2_{\mathrm{w}}$ in (3.24) at $x_{\mathrm{w}} = \infty$ is compensated by the divergence of the dilaton factor.

Like the full free energy (3.17), this expression is divergent at large $x_{\rm w}$ — it should represent the combined energy of the background dilaton field and the black hole. Subtracting the vacuum value $E_{\rm w}^{\rm (vac)}$ in (3.23) following [1] we get

$$E'_{\rm w} = E_{\rm w} - E_{\rm w}^{\rm (vac)} = 8\pi\mathcal{D}_{\rm w}T_0\left[1 - \frac{T_0}{T_{\rm w}}\left(1 + p\frac{T_0^2}{T_{\rm w}^2}\right)^{-3/2}\left(1 + \frac{1}{2}p + (p^2+p)\frac{T_0^2}{2T_{\rm w}^2}\right)\right]$$
$$= 4\pi T e^{-2\phi_0}(1-p) + O(e^{-2bx_{\rm w}})\,. \tag{3.28}$$

The asymptotic value of the subtracted energy should represent the mass of the black hole,[13]

$$M = E' = 4\pi T e^{-2\phi_0}(1-p)\,. \tag{3.29}$$

Expressing the entropy (3.26) in terms of the mass we thus get

$$S = \beta E' = \frac{M}{T}\,. \tag{3.30}$$

This is the same relation that was found in [1] for the leading-order black hole (and is the 2-d analog of the 4-d Schwarzschild black hole relation $S = 4\pi M^2$). It is, of course, consistent with the vanishing of the subtracted value of the free energy in (3.20).

3.2 "Dilaton subtraction" approach

Given that the matrix model result [2] for the tree-level free energy is finite and *non-zero*, it is natural to try to look for alternative prescriptions for computing F that may also give a non-vanishing result on the effective field theory side. Let us go back to the expression for the euclidean action (3.11) and suggest a subtraction procedure that seems naturally adapted to comparison with the matrix model calculation.

The matrix model computation uses Sine-Liouville model (see section 4 below) so the spatial direction x in (3.2) may be interpreted as a "Liouville direction" φ. Furthermore, it can be always traded for the dilaton field ϕ which is the true counterpart of the Liouville field φ.[14] Let us then re-express the effective action (3.11)

[13]This expression agrees, for $p=0$, with the ADM mass of the leading-order black hole [3, 1, 5]. One may expect that, like the free energy, the ADM mass of the exact black hole should not depend on the unknown details of the bulk term in the action, and should be given just by a boundary term. However, the expressions for the ADM mass in [3, 1, 5] (see also [34]) were derived using leading-order form of the action or equations of motion and do not seem to apply to the present exact background. In particular, the above result for M is different from what follows from the expression $M_{\rm ADM} = 2(e^{-2\phi}\partial_x f)_{x=\infty}$ in [1] for the background (3.2) and (3.3) (we correct a misprint in [1]: $f^2 \to f$; equivalent result was given in [5]). It also disagrees with the expression for the mass given in [33]. Using the definitions of the ADM mass in [1, 5] one finds $M = 4\pi T e^{-2\phi_0}(1-p)^{-3/2}$. The definition in [3], i.e. $M = 4e^{-2\phi_\infty}\partial_x\varphi$, $\phi = \phi_\infty + \varphi + O(e^{-4bx})$ (see also [34]) gives $M = 4\pi T e^{-2\phi_0}(1-p)^{-1}$. All these expressions agree for $p=0$ only.

[14]For example, one can change coordinates in (3.2) and (3.3) so that the dilaton becomes a linear function of the new coordinate x'.

JHEP06(2001)021

in terms of the dilaton variable ϕ using (3.3)

$$I_{\partial\mathcal{M}} = -\frac{4\pi}{1-p}e^{-2\phi_0}\left[-1+\sqrt{p^2+4(1-p)e^{-4(\phi-\phi_0)}}\right]. \tag{3.31}$$

This expression contains a divergence corresponding to the infrared behaviour $\phi \to -\infty$ in the target space. This divergence may be interpreted as an UV one on the world sheet of the Sine-Liouville model regarded as a non-critical string theory (if φ is associated with the conformal factor of the 2-d metric then the covariant cutoff corresponds to $e^{-2\varphi}|\Delta z|^2 > 1/\Lambda^2 \to 0$). Introducing a cutoff Λ on $e^{-\phi} \to \infty$ (similar to the cutoff used in the matrix model) we get from (3.31)

$$I_{\partial\mathcal{M}} \simeq -\frac{8\pi}{(1-p)^{1/2}}\Lambda^2 + \frac{4\pi}{1-p}e^{-2\phi_0} - \frac{\pi p}{(1-p)^{3/2}}e^{2\phi_0}\frac{1}{\Lambda^2} + O\left(\frac{1}{\Lambda^4}\right). \tag{3.32}$$

As in the matrix model approach discussed in section 4, it is natural to drop the non-universal cutoff-dependent terms in (3.32). We then obtain the following finite value for the string partition function[15]

$$-\mathcal{F}_0 \equiv I'_{\partial\mathcal{M}} = \frac{4\pi}{1-p}e^{-2\phi_0}. \tag{3.33}$$

It has the same scaling $\mathcal{F} \sim e^{-2\phi_0} \sim g_s^{-2}$ as the matrix model result. The coefficients are difficult to compare systematically: for that one needs to determine several more orders in the g_s^2 expansion.

3.3 Alternative subtractions

One may look also for other alternative subtractions that may reconcile the results of the matrix model and effective field theory approaches. One could expect that a subtracted field-theory expression for F could be zero for the leading-order $p = 0$ background but non-zero for $p \neq 0$, i.e. for the exact background. However, such a prescription which is, at the same time, consistent with thermodynamic relations does not seem to exist. Let us make few remarks about other subtraction recipes which either lead to the zero value of the energy inconsistent with thermodynamics or still give $F = 0$ for any value of p.

First, let us go back to the expansion (3.17) and subtract the divergence in the naive (non-reparametrization invariant) way, thus getting a *non-zero* result for the free energy

$$F' = -4\pi T e^{-2\phi_0}(1-p)^{-1}. \tag{3.34}$$

If one formally computes the entropy according to $S_\infty = -\partial F_{\rm w}/\partial T$ one then gets the result

$$S = 4\pi e^{-2\phi_0}(1-p)^{-1}, \tag{3.35}$$

[15] The same expression for the finite value of the action (3.9) was obtained by S. Alexandrov using a subtraction of the flat space contribution "at equal dilatons" [35].

which is different from (3.26) but is still in agreement with the leading-order expression for the entropy once we set $p = 0$: $S = 4\pi(e^{-2\phi})_{\text{horizon}} = 4\pi e^{-2\phi_0}$ [1, 6]. The problem, however, is that then $E' = F' + ST$ (see (3.22)) is zero instead of being equal to the black hole mass.

This is a general conclusion: if one considers the subtracted value of the free energy to be a finite linear function of the temperature directly at infinity and varies T then the resulting entropy is constant while the energy is zero.[16]

Another possibility is to follow [5] and modify the definition of the boundary term in the action (3.8) to make the subtraction of the infinite term automatic

$$I'_{\partial\mathcal{M}} = -2\int_0^{\beta_0} d\tau \partial_x[(f - f_\infty)e^{-2\phi}]|_{x=x_{\text{bndry}}} \,. \tag{3.36}$$

Compared to (3.36) we replaced f by $f - f_\infty$, where f_∞ is its asymptotic value, i.e. $(1-p)^{-1/2}$ in the case of (3.2). This gives, expanding for large x,

$$I'_{\partial\mathcal{M}} = -4\pi e^{-2\phi_0}\frac{1-2p}{(1-p)^2}e^{-2bx} + O(e^{-4bx})\,, \tag{3.37}$$

i.e. the subtracted value of the action and thus of the free energy is again zero for any p.

4. Free energy in matrix model of 2-d black hole

4.1 Noncritical $c_M = 1$ string theory and free energy from matrix model

The matrix quantum mechanics (MQM) description of the 2-d string theory on the black hole background [2] appeared to be possible due to the duality, proposed in [36], between the two sigma models: the $(\frac{\mathrm{SL}(2,\mathbb{C})}{\mathrm{SU}(2)\times \mathrm{U}(1)})_k$ "euclidean black hole" coset model [3] and the Sine-Liouville (SL) theory with the action (in this section α'=1)

$$\mathcal{S}_{SL} = \frac{1}{4\pi}\int d^2z\left[(\partial X)^2 + (\partial\varphi)^2 + Q\hat{\mathcal{R}}\varphi + \lambda e^{\gamma\varphi}\cos R(X_L - X_R)\right]. \tag{4.1}$$

Here the scalar field X (decomposed on shell as $X(z,\bar{z}) = X_L(\bar{z}) + X_R(z)$) is compactified on a circle of radius R and φ is the "Liouville field". $\hat{\mathcal{R}} = \sqrt{g}R^{(2)}$ is a two dimensional background curvature normalised so that $\frac{1}{4\pi}\int d^2z\hat{\mathcal{R}} = 2 - 2h$, where h is the genus of the surface.

[16] One could try to use an alternative recipe by observing that since in the present case p is related to k or b and thus to T (see (3.4) and (3.12)) one should re-express everything in terms of T before differentiating over it. In that case (3.34) would take the form (see (3.12)) $F' = -4\pi e^{-2\phi_0}\frac{T}{1-8\pi^2\alpha' T^2}$ so that the entropy would be $S = 4\pi e^{-2\phi_0}\frac{1+8\pi^2\alpha' T^2}{1-8\pi^2\alpha' T^2}$. The problem with this attempt is that for $\alpha' \to 0$ one does not reproduce the previous leading-order result for the energy (black hole mass): the energy still vanishes in the $\alpha' \to 0$ limit.

JHEP06(2001)021

The central charges of the two theories are to be equal, i.e. $c = \frac{3k}{k-2} - 1 = 1 + 6Q^2 + 1$ which gives $Q^2 = \frac{1}{k-2}$. To have the (1,1) conformal dimension of the SL perturbation term we have to put $\gamma = -Q^{-1} = -\sqrt{k-2}$ with the radius of compactification of x being

$$R = \sqrt{k}\,, \tag{4.2}$$

i.e. the same as the radius of the cigar far from the tip (see (3.2) and (3.12)).

The MQM describes a theory which is slightly different from the SL theory (4.1):

$$\mathcal{S}_{c_M=1} = \frac{1}{4\pi}\int d^2z\left[(\partial X)^2 + (\partial\varphi)^2 + 2\hat{\mathcal{R}}\varphi + \mu\varphi e^{-2\varphi} + \lambda e^{(R-2)\varphi}\cos R(X_L - X_R)\right]. \tag{4.3}$$

Here the central charge is always $c = 26$, due to the choice of the background charge $Q = 2$, whereas the compactification radius R of X is arbitrary within the interval $1 < R < 2$. This choice corresponds to the central charge $c_M = 1$ of the matter in the string theory. Note that the coupling λ plays the role of the fugacity of vortices (windings) with charges ± 1 on the world sheet, similar to the fugacity of vortices in the usual Sine-Gordon model.

However, the two theories (4.1) and (4.3) become the same for the zero value of the "cosmological constant" $\mu = 0$ (or rather in the limit $y = \mu\lambda^{\frac{2}{R-2}} \to 0$, see below) and the radius $R = 3/2$, corresponding to the level $k = 9/4$ which is precisely the string theory dilatonic black hole point in the coset description [3].

In the matrix model approach of [2] it was found that for the model (4.3) the first-quantized string partition function $\mathcal{F}(\mu, \lambda)$ (which, in space-time interpretation, is essentially the free energy, up to a constant inverse temperature factor $\beta = 2\pi R$), satisfies the Toda equation:

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\mathcal{F}(\lambda,\mu) + \exp\left[\mathcal{F}(\lambda,\mu+i) + \mathcal{F}(\lambda,\mu-i) - 2\mathcal{F}(\lambda,\mu)\right] = 1\,, \tag{4.4}$$

with the boundary condition given by the $\mu^{-1} \sim g_s$ expansion:

$$\mathcal{F}(\mu, 0) = -\frac{R}{2}\mu^2\log\frac{\mu}{\Lambda} - \frac{1}{24}(R + R^{-1})\log\frac{\mu}{\Lambda} + R\sum_{h=2}^{\infty}\mu^{-2(h-1)}\tilde{f}_h^{(R)} + O(e^{-2\pi\mu})\,. \tag{4.5}$$

Here $\tilde{f}_h^{(R)}$ are the known coefficients and Λ is a UV cutoff on the world sheet (we will drop the dependence on it below). The KPZ-DDK scaling following from the change of the couplings induced by a shift of the zero mode of the Liouville field φ imposes the constraints on the genus expansion of the free energy of the theory (4.3):

$$\mathcal{F}(\lambda,\mu) = \sum_{h=0}^{\infty}\mathcal{F}_h(\lambda,\mu)\,, \tag{4.6}$$

where

$$\mathcal{F}_0(\lambda,\mu) = \frac{R}{2-R}\mu^2\log\lambda + \lambda^{\frac{4}{2-R}}f_0(y)\,,$$

$$\mathcal{F}_1(\lambda,\mu) = -\frac{R+R^{-1}}{12(2-R)}\log\lambda + f_1(y)\,,$$
$$\mathcal{F}_h(\lambda,\mu) = (\lambda^{\frac{2}{2-R}})^{2-2h} f_h(y)\,, \qquad h\geq 2\,, \tag{4.7}$$

with

$$y \equiv \mu\lambda^{-\frac{2}{2-R}}\,.$$

The constants in front of the logarithmic terms are fixed by the boundary conditions (4.5) and the KPZ-DDK scaling.

To isolate the genus 0 (2-sphere) contribution we may take μ very large (by keeping y fixed). Then the equation (4.4) becomes:

$$\frac{1}{4}\lambda^{-1}\partial_\lambda\lambda\partial_\lambda\mathcal{F}(\lambda,\mu) + \exp\left[\partial_\mu^2\mathcal{F}(\lambda,\mu)\right] = 0. \tag{4.8}$$

From here we can immediately find the free energy in the "black hole" limit $y\to 0$ (more precisely, for the Witten's black hole case we need also to set $R = 3/2$ when (4.3) reduces to (4.1)). The relevant part of $\mathcal{F}_0$ is given by $f_0(0)$ in (4.7). Plugging the logarithmic term from the first line of (4.7) into the r.h.s. of (4.8) we find

$$\mathcal{F}_0(\lambda,0) = -C(R)\lambda^{\frac{4}{2-R}}\,, \qquad C(R) = \frac{1}{4}(2-R)^2\,. \tag{4.9}$$

The coefficient $C(R)$ can be, in principle, absorbed into the dimensionful parameter λ.[17] However, once it is chosen, it fixes all the subsequent terms in the genus expansion (4.7):

$$\mathcal{F}(\lambda,0) = -C(R)\lambda^{\frac{4}{2-R}} - \frac{R+R^{-1}}{48}\log\lambda^{\frac{4}{2-R}} + \sum_{h=2}^{\infty} f_h^{(R)}\lambda^{\frac{4-4h}{2-R}}\,. \tag{4.10}$$

Here $f_h^{(R)} = f_h(0)$ are universal functions of R different from $\tilde{f}$'s in (4.5). They can be, in principle, determined from the eq. (4.4) (this has not yet been done).

Note that not only the scaling but also the sign of this matrix model result for the free energy (4.9) is consistent with the expression (3.33) found from the "dilaton subtraction" approach in the effective field theory.

4.2 Comments on interpretation of the matrix model result

We conclude that the MQM approach to the bosonic string theory in the black hole background gives a non-zero universal genus 0 string partition function with the specific dependence on the compactification radius R.

A thermodynamic interpretation of the black hole free energy depends very much on how one changes the temperature of the system. At this stage we can take two different points of view on thermodynamical interpretation of the result (4.10):

[17] Note that in principle we have to change the sign of $\mathcal{F}_0$ to the opposite if we want to work in the physically relevant canonical ensemble (fixed N rather then fixed μ) within the MQM (see [2] for the details).

(i) following the argument of [2] we can assume that at least for $R \to 3/2$ the parameter λ is a function of R, adjusted in a way that

$$\mathcal{F}(\lambda) = 2\pi \left(R - \frac{3}{2} \right) M - \frac{R + R^{-1}}{24} \log M + \sum_{h=2}^{\infty} \bar{f}_h^{(R)} M^{-h} \,, \tag{4.11}$$

where $M \sim [\lambda(R)]^{\frac{4}{2-R}}$ is an R-independent parameter associated with the mass of the black hole. The coefficient $R - 3/2$ of the 2-sphere term is natural since it corresponds to the result of the section 3 stating that the tree-level string partition function (i.e. space-time effective action) is zero at the Hawking temperature $T = T_H = 1/3\pi$ ($T = 1/2\pi R$, see (3.12)). In that case the Hagedorn phase transition happens at this temperature, the fluctuations of the energy are strong and we have to reconsider the usual thermodynamics based on the Legendre transform, taking into account the one loop logarithmic correction (second term in the r.h.s. of (4.11)) and performing the integral over these fluctuations. The details of this point of view on thermodynamics of the 2-d black hole and some of its physical consequences can be found in [2].

(ii) we may try to adopt another point of view which seems more appropriate in the context of the 2-d string theory model (4.3) or its MQM counterpart. Namely, we may take the R-dependence of the power of λ in (4.9) seriously (the R-dependence of the coefficient $C(R)$ will not influence the following arguments) and view λ as R-independent free parameter of the theory. Let us consider the partition function of an ensemble with fixed and equal numbers n of vortices and anti-vortices of charges ± 1, respectively, which is represented by the contour integral (around zero):

$$\exp \mathcal{F}(n) = \oint \frac{d\lambda}{2\pi i \lambda^{2n+1}} \exp \mathcal{F}_0(\lambda, 0) \,, \tag{4.12}$$

where $\mathcal{F}_0(\lambda, 0)$ is given by (4.9). n shoud be even for the charge neutrality but it does not matter for large values of n we shall consider here. The saddle point calculation (justified for large n) gives:

$$\mathcal{F}(n) = (2 - R) n \log \frac{\Lambda}{\sqrt{n}} + O(n) \,, \tag{4.13}$$

where we restored the dependence on the cutoff Λ resulting from the non-universal terms in the MQM free energy.

It is natural to associate the R-dependent term in (4.13) with the mass term βM

$$M = \frac{1}{2\pi} n \log \frac{\Lambda}{\sqrt{n}} \,, \tag{4.14}$$

while the R-independent term - with the entropy S:

$$S = 2n \log \frac{\Lambda}{\sqrt{n}} \,. \tag{4.15}$$

We see that the free energy now vanishes not at the Hawking temperature $R = 3/2$, as in (4.11) , but rather at the Kosterlitz-Thouless temperature $T_{KT} = 1/2\pi R_{KT}$, $R_{KT} = 2$.

These formulas predict that at the Hawking temperature corresponding to $R = 3/2$ the free energy $F = -T\mathcal{F}$ and the entropy are given by

$$F = -\frac{1}{3} M \,, \qquad S = 4\pi M \,. \tag{4.16}$$

The finiteness and string coupling scaling of F are consistent with the result (3.33) following from the effective action approach (although they differ by sign in the ensemble with fixed n). The expression for S is different from $S = 3\pi M$ implied by (3.30) at $T = T_H = 1/3\pi$.

As a consequence of the FZZ conjecture [36] the entropy (4.15) counts the number of states of the 2-d black hole. This picture can be made quite precise in the MQM language since we can explicitly write down the hamiltonian $\hat{H}_n$ of evolution in imaginary time $\beta = 2\pi R$ in the sector with fixed (conserved) number n of vortex - anti-vortex pairs (see appendix). The Gibbs partition function of MQM is defined as

$$e^{\mathcal{F}(n)} = \mathrm{Tr}_n \, e^{-2\pi R \hat{H}_n} \,, \tag{4.17}$$

where the trace is taken over the Hilbert space of MQM states belonging to a reducible representation describing n vortex - anti-vortex pairs. Our estimate for $\mathcal{F}_n$ is given by the formula (4.13) extracted from the Toda equation. We see that the non-singlet states have a large entropy $\sim M$ typical of a black hole. It would be very instructive to derive it directly from the hamiltonian given in appendix.

As we know from the collective field theory approach [37] to the singlet sector of the MQM, such a hamiltonian should produce an effective action for the tachyon scattering amplitudes. This would be the most direct way to probe the background of this string theory.

Let us stress that we do not want to insist on this second point of view as the only possible approach to thermodynamics of the 2-d black hole. The above definition of the temperature is natural only in the context of the model (4.3) or its MQM counterpart, since they have a specific dependence of the string coupling on R. When we vary the radius R (i.e. the inverse temperature) in this model we do not add or remove any new matter fields from the system since the matter central charge is always fixed to be $c_M = 1$, while as in the coset WZW model (and the Sine-Liouville model (4.1)) we change c_M when changing R, as is clear from (4.2). Also, the mass of the black hole in the coset WZW model does not a priori depend on R.

To prefer one of these two options we would have to perform a truely microscopic calculation of thermodynamic properties of the system based on the MQM hamiltonian.

5. Concluding remarks

In this paper we tried to address some not yet well understood issues concerning thermodynamic interpretation of the 2-d black hole in string theory. The main purpose was to try to reconcile the old result of the effective action approach to this problem [1] (generalizing it to the case of the exact black hole [8]) and the new result following from the matrix model approach [2].

We have shown that starting with the exact conformal 2-d black hole background (3.2) and (3.3) and subtracting the dilatonic vacuum contribution from the free energy following the procedure of [1] one finds the vanishing result. At the same time, the matrix model calculation of [2] gave a non-zero result for the seemingly the same quantity. One possibility to explain this discrepancy was suggested in [2]: the non-vanishing matrix theory result for the tree-level free energy should be interpreted as a non-universal contribution which is to be omitted in comparing with space-time theory.

Hovever, the matrix theory free energy given by the solution of the Toda equation is automatically finite and looks rather universal: the non-universal IR divergent terms are already absent there. This is unlike the effective field theory expression containing the divergent dilaton vacuum contribution (3.17). The difference is obviously in how the subtraction of the divergent vacuum contribution is done at the effective string field theory and the matrix theory sides. We suggested that the string partition function in the matrix model is a different quantity from the one given by the free energy (3.19) with the "subtraction at the wall" prescription of [1] — the corresponding effective action should instead be defined by the "dilaton subtraction" prescription of section 3.2. The advantage of the latter recipe is that it is not related to any specific definition of the temperature (which plays an important role in the approach of [1]).[18]

We have also proposed in section 4.2 a "microscopic" estimate for the entropy and the mass of the black hole based on fact that the Toda equation solution in the matrix

[18] It was conjectured in [2] that in the matrix theory analog of the fermionic string (assuming it exists) the expression for the sphere contribution to the free energy should vanish. That would be in agreement with the vanishing of the "subtracted at the wall" free energy on the string theory side (note that the exact black hole background in the supersymmetric case is given by (3.1) with $D-10+4\alpha' b^2=0$, $\alpha' b^2=1/k$). However, the dilaton divergence subtraction of section 3.2 would still give a non-vanishing result for the free energy in this case. A resolution of this contradiction should await an actual construction of fermionic string analog of MQM.

model can be interpreted as a calculation of the ground state of the matrix hamiltonian in the relevant non-singlet representation of the $\mathrm{U}(N)$ symmetry of the model.

It remains an open and interesting problem of how to derive the boundary terms (2.7) in string theory effective action and thus the black hole entropy (cf. [38]) directly from the string sigma model path integral on 2-sphere.[19] The local equations of motion (beta function conditions of conformal invariance) do not "know" about the boundary terms in the effective action, so this is the case which highlights the fundamental role of the string sigma model partition function.

It is clear that we are still far from understanding of the general picture explaining the thermodynamic behaviour of the 2-d black hole. Recent progress based on the matrix model formulation gives hopes for a truely microscopic description of this interesting model.

Acknowledgments

We are grateful to S. Alexandrov, A. Peet, J. Polchinski and S. Shenker for useful discussions. Part of this work was done while we were participating in M-theory program at ITP, Santa Barbara supported by the NSF grant PHY99-07949. The work of V.A.K. is partially supported by the European Commission RTN programme HPRN-CT-2000-00122. The work of A.A.T. is partially supported by the DOE grant DE-FG02-91ER40690 and also by PPARC SPG 00613, INTAS 991590 and the EC RTN HPRN-CT-2000-00131 grants.

A. Hamiltonian of the matrix model in non-singlet representation

In the matrix model approach of [2, 39] the model (4.3) is described by the MQM hamiltonian

$$\hat{H} = -\frac{1}{2N}\Delta_X - \frac{1}{2}\,\mathrm{tr}\,X^2\,,$$

where X is an $N \times N$ hermitean matrix coordinate and Δ_X is a laplacian on the space of hermitean matrices. The model has the $\mathrm{SU}(N)$ invariance and hence the states can be classified according to representations of $\mathrm{SU}(N)$. For an arbitrary representation r the hamiltonian $\hat{H}_r$ can be expressed in terms of the eigenvalues $z_1, \ldots, z_N$ of the matrix X

$$H_r = P_r \sum_{k=1}^{N}\left[-\frac{1}{2N}\frac{\partial^2}{\partial z_k^2} - \frac{N}{2}z_k^2\right] + \frac{1}{4N}\sum_{i<j}\frac{\widehat{\mathcal{T}}^r_{ij}\widehat{\mathcal{T}}^r_{ij}}{(z_i - z_j)^2}\,. \tag{A.1}$$

[19] It may be useful to try to follow the analogy with the particle theory path integral representation for the logarithm of determinant of a laplacian. Indeed, the local cutoff-dependent part of it is given by the Seeley coefficients containing bulk and boundary terms.

P_r is a projector onto the subspace of all zero weight vectors in the space of representation (i.e. the kernel space of the generators of the Cartan subalgebra) and $\widehat{\tau}^r_{ij}$ are the N^2 generators in this representation (see [39, 40] for details).

The (reducible) representation describing the dynamics of n vortex - anti-vortex pairs in the system corresponds to the character

$$\chi_n(\Omega) = (\mathrm{tr}\,\Omega)^n (\mathrm{tr}\,\Omega^\dagger)^n\,,$$

where $\Omega \in \mathrm{Adj}[\mathrm{SU}(N)]$ (see [39] and [2] for the explanation of the connection between representations and vortex - anti-vortex configurations). This representation (which we shall denote using index n instead of general label r) is just a direct product of n fundamental and n anti-fundamental representations. The dimension of the representation is N^{2n}, the projector P_n is

$$P_n = \big(\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n}, \underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n}\big)\,, \tag{A.2}$$

where $\mathbb{1}$ is the $N \times N$ unity matrix and the generators are defined by the following action on an arbitrary $N \times N$ matrix A

$$\begin{aligned}\sum_{i,j=1}^{N} \widehat{\tau}^n_{ij} A_{ij} = \sum_{m=1}^{n} \Big[&\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{m-1}\times A\times\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n-m}\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n} - \\ &-\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n}\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{m-1}\times A\times\underbrace{\mathbb{1}\times,\ldots,\times\mathbb{1}}_{n-m}\Big]\,.\end{aligned} \tag{A.3}$$

References

[1] G.W. Gibbons and M.J. Perry, *The physics of 2-d stringy space-times*, *Int. J. Mod. Phys.* **D1** (1992) 335 [hep-th/9204090].

[2] V. Kazakov, I.K. Kostov and D. Kutasov, *A matrix model for the two dimensional black hole*, hep-th/0101011.

[3] E. Witten, *On string theory and black holes*, *Phys. Rev.* **D 44** (1991) 314.

[4] S. Elitzur, A. Forge and E. Rabinovici, *Some global aspects of string compactifications*, *Nucl. Phys.* **B 359** (1991) 581;
G. Mandal, A.M. Sengupta and S.R. Wadia, *Classical solutions of two-dimensional string theory*, *Mod. Phys. Lett.* **A 6** (1991) 1685.

[5] C.R. Nappi and A. Pasquinucci, *Thermodynamics of two-dimensional black holes*, *Mod. Phys. Lett.* **A 7** (1992) 3337 [gr-qc/9208002].

[6] V.P. Frolov, *Two-dimensional black hole physics*, *Phys. Rev.* **D 46** (1992) 5383.

[7] J.D. Creighton and R.B. Mann, *Quasilocal thermodynamics of two-dimensional black holes*, *Phys. Rev.* **D 54** (1996) 7476;
R.C. Myers, *Black hole entropy in two-dimensions*, *Phys. Rev.* **D 50** (1994) 6412 [hep-th/9405162];
S.N. Solodukhin, *Two-dimensional quantum corrected eternal black hole*, *Phys. Rev.* **D 53** (1996) 824 [hep-th/9506206].

[8] R. Dijkgraaf, H. Verlinde and E. Verlinde, *String propagation in a black hole geometry*, *Nucl. Phys.* **B 371** (1992) 269.

[9] I. Bars and K. Sfetsos, *Conformally exact metric and dilaton in string theory on curved space-time*, *Phys. Rev.* **D 46** (1992) 4510 [hep-th/9206006]; *Exact effective action and space-time geometry N gauged WZW models*, *Phys. Rev.* **D 48** (1993) 844 [hep-th/9301047].

[10] A.A. Tseytlin, *Effective action of gauged wzw model and exact string solutions*, *Nucl. Phys.* **B 399** (1993) 601 [hep-th/9301015];
Conformal sigma models corresponding to gauged Wess-Zumino-Witten theories, *Nucl. Phys.* **B 411** (1994) 509 [hep-th/9302083].

[11] A.A. Tseytlin, *On the form of the black hole solution in $D = 2$ theory*, *Phys. Lett.* **B 268** (1991) 175.

[12] I. Jack, D.R. T. Jones and J. Panvel, *Exact bosonic and supersymmetric string black hole solutions*, *Nucl. Phys.* **B 393** (1993) 95 [hep-th/9201039].

[13] A.A. Tseytlin, *On field redefinitions and exact solutions in string theory*, *Phys. Lett.* **B 317** (1993) 559 [hep-th/9308042].

[14] A.A. Tseytlin, *Exact solutions of closed string theory*, *Class. and Quant. Grav.* **12** (1995) 2365 [hep-th/9505052].

[15] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D 15** (1977) 2752.

[16] C.G. Callan, R.C. Myers and M.J. Perry, *Black holes in string theory*, *Nucl. Phys.* **B 311** (1989) 673;
R. Myers, *Superstring gravity and black holes*, *Nucl. Phys.* **B 289** (1987) 701.

[17] M. Visser, *Dirty black holes: entropy as a surface term*, *Phys. Rev.* **D 48** (1993) 5697 [hep-th/9307194];
V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [gr-qc/9403028].

[18] T. Jacobson, G. Kang and R.C. Myers, *On black hole entropy*, *Phys. Rev.* **D 49** (1994) 6587 [gr-qc/9312023]; *Increase of black hole entropy in higher curvature gravity*, *Phys. Rev.* **D 52** (1995) 3518 [gr-qc/9503020];
G.L. Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Supersymmetric black hole*

solutions with R^2 interactions, hep-th/0003157;
S.O. Alexeev and M.V. Pomazanov, *Black hole solutions with dilatonic hair in higher curvature gravity*, *Phys. Rev.* **D 55** (1997) 2110 [hep-th/9605106];
R.C. Myers, *Black holes in higher curvature gravity*, gr-qc/9811042.

[19] D.V. Fursaev and S.N. Solodukhin, *On the description of the riemannian geometry in the presence of conical defects*, *Phys. Rev.* **D 52** (1995) 2133 [hep-th/9501127].

[20] E.S. Fradkin and A.A. Tseytlin, *Quantum string theory effective action*, *Nucl. Phys.* **B 261** (1985) 1.

[21] C. Lovelace, *Stability of string vacua, 1. A new picture of the renormalization group*, *Nucl. Phys.* **B 273** (1986) 413;
B.E. Fridling and A. Jevicki, *Nonlinear sigma models as S matrix generating functionals of strings*, *Phys. Lett.* **B 174** (1986) 75;
A.A. Tseytlin, *Vector field effective action in the open superstring theory*, *Nucl. Phys.* **B 276** (1986) 391.

[22] A.A. Tseytlin, *Mobius infinity subtraction and effective action in sigma model approach to closed string theory*, *Phys. Lett.* **B 208** (1988) 221.

[23] A.A. Tseytlin, *Renormalization of mobius infinities and partition function representation for string theory effective action*, *Phys. Lett.* **B 202** (1988) 81.

[24] J. Liu and J. Polchinski, *Renormalization of the mobius volume*, *Phys. Lett.* **B 203** (1988) 39.

[25] A.A. Tseytlin, *Conditions of weyl invariance of two-dimensional sigma model from equations of stationarity of 'central charge' action*, *Phys. Lett.* **B 194** (1987) 63.

[26] C.G. Callan, E.J. Martinec, M.J. Perry and D. Friedan, *Strings in background fields*, *Nucl. Phys.* **B 262** (1985) 593;
A.A. Tseytlin, *Sigma model Weyl invariance conditions and string equations of motion*, *Nucl. Phys.* **B 294** (1987) 383;
R.R. Metsaev and A.A. Tseytlin, *Order α' (two loop) equivalence of the string equations of motion and the sigma model Weyl invariance conditions: dependence on the dilaton and the antisymmetric tensor*, *Nucl. Phys.* **B 293** (1987) 385;
C.M. Hull and P.K. Townsend, *String effective actions from sigma model conformal anomalies*, *Nucl. Phys.* **B 301** (1988) 197;
H. Osborn, *String theory effective actions from bosonic sigma models*, *Nucl. Phys.* **B 308** (1988) 629.

[27] J.W. York, *Role of conformal three geometry in the dynamics of gravitation*, *Phys. Rev. Lett.* **28** (1972) 1082.

[28] R.C. Myers, *Higher derivative gravity, surface terms and string theory*, *Phys. Rev.* **D 36** (1987) 392.

JHEP06(2001)021

[29] R.E. Kallosh, O.V. Tarasov and I.V. Tyutin, *One loop finiteness of quantum gravity off mass shell*, *Nucl. Phys.* **B 137** (1978) 145.

[30] A.A. Tseytlin, *Ambiguity in the effective action in string theories*, *Phys. Lett.* **B 176** (1986) 92.

[31] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, *Nucl. Phys.* **B 277** (1986) 1.

[32] G.T. Horowitz and A.A. Tseytlin, *On exact solutions and singularities in string theory*, *Phys. Rev.* **D 50** (1994) 5204 [hep-th/9406067].

[33] M.J. Perry and E. Teo, *Nonsingularity of the exact two-dimensional string black hole*, *Phys. Rev. Lett.* **70** (1993) 2669 [hep-th/9302037].

[34] H. Liebl, D.V. Vassilevich and S. Alexandrov, *Hawking radiation and masses in generalized dilaton theories*, *Class. and Quant. Grav.* **14** (1997) 889 [gr-qc/9605044].

[35] S.Yu. Alexandrov, private communication.

[36] V. Fateev, A. Zamolodchikov and Al. Zamolodchikov, unpublished.

[37] S.R. Das and A. Jevicki, *String field theory and physical interpretation of $D = 1$ strings*, *Mod. Phys. Lett.* **A 5** (1990) 1639;
J. Polchinski, *What is string theory?*, hep-th/9411028.

[38] L. Susskind and J. Uglum, *Black hole entropy in canonical quantum gravity and superstring theory*, *Phys. Rev.* **D 50** (1994) 2700 [hep-th/9401070].

[39] D. Boulatov and V. Kazakov, *One-dimensional string theory with vortices as the upside down matrix oscillator*, *Int. J. Mod. Phys.* **A 8** (1993) 809 [hep-th/0012228].

[40] V.A. Kazakov, *Solvable matrix models*, hep-th/0003064.

[41] G.L. Cardoso, B. de Wit and T. Mohaupt, *Area law corrections from state counting and supergravity*, *Class. and Quant. Grav.* **17** (2000) 1007 [hep-th/9910179].

JHEP06(2001)021